

CONJUGACY BETWEEN POLYNOMIAL BASINS

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ABSTRACT. In this article, we study the properties of conjugacies between polynomial basins. For any conjugacy, there is a quasiconformal conjugacy in the same homotopy class minimizing the dilatation. We compute the precise value of the minimal dilatation. The quasiconformal conjugacy minimizing the dilatation is not unique in general. We give a necessary and sufficient condition when the extremal map is unique.

1. INTRODUCTION

Let f be a polynomial of degree $d \geq 2$. The complex plane can be decomposed into the union of its open and connected *basin of infinity*

$$X(f) := \{z \in \mathbb{C} : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}$$

and its complement, the compact *filled Julia set* $K(f)$. The *Green function* $G_f : \mathbb{C} \rightarrow [0, \infty)$ of f is given by

$$G_f(z) = \lim_{n \rightarrow \infty} d^{-n} \log^+ |f^n(z)|,$$

where $\log^+(t) = \max\{0, \log t\}$ for $t \in [0, \infty)$. The function G_f vanishes exactly on $K(f)$, harmonic on $X(f)$, continuous on \mathbb{C} , and satisfies $G_f(f(z)) = dG_f(z)$ for all $z \in \mathbb{C}$.

We denote by $C(f)$ the critical set of f in \mathbb{C} , and $P(f) := \overline{\cup_{k \geq 1} f^k(C(f))}$ the postcritical set. Consider the dynamic of f restricted in $X(f)$. We say that ϕ is a conjugacy between $(f, X(f))$ and $(g, X(g))$ if $\phi : X(f) \rightarrow X(g)$ is a homeomorphism such that $\phi \circ f = g \circ \phi$ in $X(f)$. It determines a homotopy class $[\phi]$ consisting of all conjugacies between $(f, X(f))$ and $(g, X(g))$, homotopic to ϕ rel the set $P(f) \cap X(f)$. It's known that the homotopy class $[\phi]$ always contains a quasiconformal (or smooth) map.

Let ϕ be a quasiconformal conjugacy between $(f, X(f))$ and $(g, X(g))$, let

$$\mu_\phi = \frac{\bar{\partial}\phi}{\partial\phi} = \frac{\phi_{\bar{z}} d\bar{z}}{\phi_z dz}$$

be its Beltrami form. The esssup norm $\|\mu_\phi\|$ of μ_ϕ measures the dilatation of ϕ . Let $\delta_{[\phi]} = \inf\{\|\mu_\psi\| : \psi \in [\phi] \text{ is a quasiconformal map}\}$. Note that the

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quasiconformal maps in $[\phi]$ form a normal family, there is a quasiconformal map $\varphi \in [\phi]$ whose dilatation equals $\delta_{[\phi]}$. The natural questions are:

Suppose ϕ is a conjugacy between $(f, X(f))$ and $(g, X(g))$, what is $\delta_{[\phi]}$? Further, let $\varphi \in [\phi]$ with $\|\mu_\varphi\| = \delta_{[\phi]}$, when is φ unique?

This note is devoted to answer these questions. We compute the precise value $\delta_{[\phi]}$ and give a necessary and sufficient condition when the extremal map is unique. Since some basic properties of conjugacies are needed first, we state the result in the next section. The main result is Theorem 2.6.

This work is inspired by DeMarco and Pilgrim's sequel works [DP1, DP2, DP3]. The notations we use here are borrowed from their's. The original idea concerning the deformation of polynomial basins comes from the *wring motion* (also called the *Branner-Hubbard motion*) introduced by Branner and Hubbard [BH], and the Teichmüller theory of rational maps developed by McMullen and Sullivan [MS]. The uniqueness of the extremal quasiconformal conjugacy follows from a result of Reich and Strebel [RS]. The theory of extremal quasiconformal mappings is used by Cui [C] to construct the extremal conjugacy between rational maps.

We denote by \mathbb{D} the unit disk, $\mathbb{H} = \{\Re z > 0\}$ the right half plane endowed the metric $|dz|/\Re z$.

2. BASIC PROPERTIES OF CONJUGACY

Let \mathcal{P}_d be the space of monic and centered polynomials. That is each $f \in \mathcal{P}_d$ takes the form

$$f(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_1z + a_0,$$

where $a_{d-2}, \dots, a_0 \in \mathbb{C}$. Any polynomial of degree d is conjugate by an automorphism of \mathbb{C} to some element of \mathcal{P}_d .

Set $M(f) = \max\{G_f(c) : c \in C(f)\}$. The Böttcher map ϕ_f of f is defined near ∞ with $\phi'_f(\infty) = 1$ and it can be extended to an isomorphism (c.f. [M])

$$\phi_f : \{z \in \mathbb{C} : G_f(z) > M(f)\} \rightarrow \{w \in \mathbb{C} : |w| > e^{M(f)}\}.$$

2.1. Conjugacy preserves escape order.

Lemma 2.1. *Let ϕ be a conjugacy between $(f, X(f))$ and $(g, X(g))$, then for any $z_1, z_2 \in X(f)$,*

1. $G_f(z_1) > G_f(z_2)$ if and only if $G_g(\phi(z_1)) > G_g(\phi(z_2))$.
2. If ϕ is quasiconformal, then

$$\frac{G_f(z_1) - G_f(z_2)}{K(\phi)} \leq G_g(\phi(z_1)) - G_g(\phi(z_2)) \leq K(\phi)(G_f(z_1) - G_f(z_2)).$$

Proof. We assume $G_f(z_1) > G_f(z_2)$. Choose $k \geq 0$ such that $G_f(f^k(z_2)) \geq M(f)$, then the set $A = \{z \in \mathbb{C} : G_f(f^k(z_2)) < G_f(z) < G_f(f^k(z_1))\}$ is an

annulus. Since $\text{mod}(\phi_g g^n \phi(A)) = \text{mod}(g^n \phi(A)) = d^n \text{mod}(\phi(A)) \rightarrow \infty$ as $n \rightarrow \infty$, we conclude by ([Mc], Theorem 2.1) that when n is large enough,

$$\max_{p \in \alpha_n} |p| < \min_{q \in \beta_n} |q|,$$

where α_n and β_n are inner and outer boundaries of $\phi_g g^n \phi(A)$, respectively. Since $\phi_g g^{n+k}(\phi(z_1)) \in \beta_n$ and $\phi_g g^{n+k}(\phi(z_2)) \in \alpha_n$, we have that $|\phi_g g^{n+k}(\phi(z_1))| > |\phi_g g^{n+k}(\phi(z_2))|$. Thus $G_g(\phi(z_1)) > G_g(\phi(z_2))$. The ‘if’ part follows from the same argument, by considering ϕ^{-1} .

To prove the bi-Lipschitz inequality when ϕ is quasiconformal, we first observe that $G_f(z_1) = G_f(z_2)$ if and only if $G_g(\phi(z_1)) = G_g(\phi(z_2))$ by the above conclusion. We may still assume $G_f(z_1) > G_f(z_2)$, then

$$\phi(A) = \{w \in \mathbb{C} : G_g(g^k(\phi(z_2))) < G_g(w) < G_g(g^k(\phi(z_1)))\}.$$

Note that $2\pi \text{mod}(\phi(A)) = d^k(G_g(\phi(z_1)) - G_g(\phi(z_2)))$ and $2\pi \text{mod}(A) = d^k(G_f(z_1) - G_f(z_2))$. By the modular distortion

$$K(\phi)^{-1} \text{mod}(A) \leq \text{mod}(\phi(A)) \leq K(\phi) \text{mod}(A),$$

we get the required inequality. \square

Given a polynomial $f \in \mathcal{P}_d$, the *DeMarco-McMullen tree* $T(f)$ of f is the quotient of $X(f)$ obtained by collapsing each connected component of a level set of G_f to a single point, see [DM]. Let $\pi_f : X(f) \rightarrow T(f)$ be the quotient map, then f induces a tree map $\sigma_f : T(f) \rightarrow T(f)$ by $\sigma_f \circ \pi_f = \pi_f \circ f$. The function G_f descends to a height function $h_f : T(f) \rightarrow \mathbb{R}_+$ by $G_f = h_f \circ \pi_f$, which induces a metric d_f on $T(f)$. A consequence of Lemma 2.1 is

Corollary 2.2. *A conjugacy ϕ between $(f, X(f))$ and $(g, X(g))$ induces an isomorphism $\iota_\phi : (T(f), \sigma_f, d_f) \rightarrow (T(g), \sigma_g, d_g)$ preserving the tree dynamics. If ϕ is quasiconformal, then ι_ϕ is $K(\phi)$ -isometric:*

$$\frac{d_f(x, y)}{K(\phi)} \leq d_g(\iota_\phi(x), \iota_\phi(y)) \leq K(\phi) d_f(x, y), \quad \forall x, y \in T(f).$$

2.2. Conjugacy preserves angular difference. The 1-form $\omega_f = 2\partial G_f = 2\frac{\partial G_f}{\partial z} dz$ satisfies $f^* \omega_f = d\omega_f$ and it provides a dynamically determined conformal metric $|\omega_f|$ on $X(f)$, with singularities at the escaping critical points and their inverse preimages (c.f. [DM, DP2, DP3]). In the metric $|\omega_f|$, for any $L > 0$, we have

$$\int_{G_f=L} |\omega_f| = 2\pi.$$

Lemma 2.3. *Let ϕ be a conjugacy between $(f, X(f))$ and $(g, X(g))$. Then for any $L > 0$ and any connected arc γ on the level set $\{G_f = L\}$, we have*

$$\int_\gamma |\omega_f| = \int_{\phi(\gamma)} |\omega_g|.$$

Proof. For $k \geq 0$, let $I_k(f, \gamma)$ (resp. $I_k(g, \phi(\gamma))$) be the integer part of $\frac{1}{2\pi} \int_{\gamma} |(f^k)^* \omega_f|$ (resp. $\frac{1}{2\pi} \int_{\phi(\gamma)} |(g^k)^* \omega_g|$). Since ϕ is a conjugacy, we have $I_k(f, \gamma) = I_k(g, \phi(\gamma))$ for all $k \geq 0$. Thus

$$\int_{\gamma} |\omega_f| = \lim_{k \rightarrow \infty} \frac{2\pi I_k(f, \gamma)}{d^k} = \lim_{k \rightarrow \infty} \frac{2\pi I_k(g, \phi(\gamma))}{d^k} = \int_{\phi(\gamma)} |\omega_g|.$$

□

2.3. Branner-Hubbard motion. In the rest of the paper, we assume $C(f) \cap X(f) \neq \emptyset$. In that case $M(f) > 0$. The fundamental annulus is $A(f) = \{z \in \mathbb{C} : M(f) < G_f(z) \leq dM(f)\}$. Set $\ell_0 = M(f)$. The critical orbits meet $A(f)$ at the level sets $\{G_f = \ell_j\}, 1 \leq j \leq N$ with $\ell_0 < \ell_1 < \dots < \ell_N = d\ell_0$. These level sets decompose $A(f)$ into N subannuli $A_j = \{z \in \mathbb{C} : \ell_{j-1} < G_f(z) < \ell_j\}, 1 \leq j \leq N$. Note that some level set $\{G_f = \ell_k\}$ may meet at least two critical orbits.

By Lemma 2.1, any conjugacy ϕ between $(f, X(f))$ and $(g, X(g))$ can induce a map $S_{\phi} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $S_{\phi} \circ G_f = G_g \circ \phi$ on $X(f)$. It is monotone increasing and satisfies $S_{\phi}(dt) = dS_{\phi}(t)$ for all $t > 0$.

We remark that by identifying t and dt in \mathbb{R}_+ , the map S_{ϕ} descends to a circle homeomorphism $\tau_{\phi} : \mathbb{S} \rightarrow \mathbb{S}$.

By Lemma 2.3, when $t > M(f)$, the angular difference between $\phi_f(z)$ and $\phi_g(\phi(z))$ is a constant for all $z \in \{G_f = t\}$. Thus ϕ induces a map $T_{\phi} : (M(f), +\infty) \rightarrow \mathbb{R}$ such that $T_{\phi}(G_f(z)) = \arg \phi_g(\phi(z)) - \arg \phi_f(z)$ when $G_f(z) > M(f)$. The map T_{ϕ} satisfies $T_{\phi}(dt) = dT_{\phi}(t)$ for all $t > M(f)$. We can extend this map to a map from \mathbb{R}_+ to \mathbb{R} , still denoted by T_{ϕ} , such that $T_{\phi}(dt) = dT_{\phi}(t)$ for all $t > 0$. Following Branner and Hubbard [BH], we call S_{ϕ} the *stretching* part of ϕ and T_{ϕ} the *turning* part of ϕ .

Note that for any two conjugacies ϕ, φ between $(f, X(f))$ and $(g, X(g))$, one has $S_{\phi}(\ell_j) = S_{\varphi}(\ell_j), \forall 1 \leq j \leq N$. So the homotopy classes of conjugacies are determined by their turning parts. The following fact is immediate:

Lemma 2.4. *Let ϕ, φ be two conjugacies between $(f, X(f))$ and $(g, X(g))$. Then $[\phi] = [\varphi]$ if and only if*

$$T_{\phi}(\ell_j) = T_{\varphi}(\ell_j), \forall 1 \leq j \leq N.$$

In fact $T_{\phi}(\ell_j) - T_{\varphi}(\ell_j) \in \frac{2\pi}{k_j} \mathbb{Z}$ for some integer $k_j \geq 1$, which is the fold of symmetry of the level set $\{G_f = \ell_j\}$.

In the following, we assume ϕ is a quasiconformal conjugacy between $(f, X(f))$ and $(g, X(g))$, then ϕ is differentiable almost everywhere in $X(f)$. This implies $S'_{\phi}(t)$ and $T'_{\phi}(t)$ exist for a.e $t \in \mathbb{R}_+$. By Lemma 2.1, we have $K(\phi)^{-1} \leq S'_{\phi}(t) \leq K(\phi)$ a.e $t \in \mathbb{R}_+$.

Lemma 2.5. *Let ϕ be a quasiconformal conjugacy between $(f, X(f))$ and $(g, X(g))$. Then we have*

$$\frac{\bar{\partial}\phi}{\partial\phi} = \frac{S'_{\phi} \circ G_f + iT'_{\phi} \circ G_f - 1}{S'_{\phi} \circ G_f + iT'_{\phi} \circ G_f + 1} \frac{\bar{\partial}G_f}{\partial G_f}.$$

Proof. It suffices to verify the equation in $\{z \in \mathbb{C} : G_f(z) > M(f)\}$ since both sides are f -invariant.

When $G_f(z) > M(f)$, we have

$$\phi_g(\phi(z)) = e^{S_\phi(G_f(z)) + iT_\phi(G_f(z))} \frac{\phi_f(z)}{|\phi_f(z)|}.$$

Set $E(z) = e^{S_\phi(G_f(z)) + iT_\phi(G_f(z))}$, then

$$\begin{aligned} \frac{\partial(\phi_g \circ \phi)}{\partial \bar{z}} &= E(z) \left[(S'_\phi(G_f(z)) + iT'_\phi(G_f(z))) \frac{\phi_f(z)}{|\phi_f(z)|} \frac{\partial G_f}{\partial \bar{z}} + \frac{\partial}{\partial \bar{z}} \left(\frac{\phi_f(z)}{|\phi_f(z)|} \right) \right], \\ \frac{\partial(\phi_g \circ \phi)}{\partial z} &= E(z) \left[(S'_\phi(G_f(z)) + iT'_\phi(G_f(z))) \frac{\phi_f(z)}{|\phi_f(z)|} \frac{\partial G_f}{\partial z} + \frac{\partial}{\partial z} \left(\frac{\phi_f(z)}{|\phi_f(z)|} \right) \right]. \end{aligned}$$

By calculation,

$$\frac{\partial}{\partial \bar{z}} \left(\frac{\phi_f(z)}{|\phi_f(z)|} \right) = -\frac{\phi_f(z)}{|\phi_f(z)|} \frac{\partial G_f(z)}{\partial \bar{z}}, \quad \frac{\partial}{\partial z} \left(\frac{\phi_f(z)}{|\phi_f(z)|} \right) = \frac{\phi_f(z)}{|\phi_f(z)|} \frac{\partial G_f(z)}{\partial z}.$$

We have

$$\frac{\bar{\partial} \phi}{\partial \phi} = \frac{\bar{\partial}(\phi_g \circ \phi)}{\partial(\phi_g \circ \phi)} = \frac{S'_\phi(G_f(z)) + iT'_\phi(G_f(z)) - 1}{S'_\phi(G_f(z)) + iT'_\phi(G_f(z)) + 1} \frac{\bar{\partial} G_f}{\partial G_f}.$$

□

2.4. Minimal dilatation and uniqueness. Let ϕ be the conjugacy between $(f, X(f))$ and $(g, X(g))$. Set

$$\tau_j^* = \frac{S_\phi(\ell_j) - S_\phi(\ell_{j-1})}{\ell_j - \ell_{j-1}} + i \frac{T_\phi(\ell_j) - T_\phi(\ell_{j-1})}{\ell_j - \ell_{j-1}}, \quad \delta_{[\phi]}^j = \left| \frac{\tau_j^* - 1}{\tau_j^* + 1} \right|, \quad j = 1, \dots, N.$$

Note that τ_j^* depends only on the homotopy class $[\phi]$.

Theorem 2.6. *Let ϕ be a conjugacy between $(f, X(f))$ and $(g, X(g))$. Then*

$$\delta_{[\phi]} = \max_{1 \leq j \leq N} \delta_{[\phi]}^j.$$

The extremal quasiconformal conjugacy is unique if and only if

$$\delta_{[\phi]}^1 = \dots = \delta_{[\phi]}^N.$$

To prove Theorem 2.6, we need the following result:

Theorem 2.7 (Reich-Strebel). *Let $\phi : R_1 \rightarrow R_2$ be a quasiconformal map between hyperbolic Riemann surfaces. If there exist a holomorphic quadratic differential q on R_1 with $\int_{R_1} |q| < +\infty$ and a number $0 \leq k < 1$, such that $\mu_\phi = k\bar{q}/|q|$, then every quasiconformal map $\psi : R_1 \rightarrow R_2$, $\psi \neq \phi$ homotopic to ϕ modulo the boundary satisfies $\|\mu_\psi\| > \|\mu_\phi\|$.*

The proof is as in [RS], p.380.

Proof of Theorem 2.6. Note that every map $\psi \in [\phi]$ is determined by the restrictions of S_ψ, T_ψ in $[\ell_0, \ell_N]$.

We construct a quasiconformal conjugacy $\psi \in [\phi]$ such that

$$(S_\psi(\ell_j), T_\psi(\ell_j)) = (S_\phi(\ell_j), T_\phi(\ell_j)), j = 0, \dots, N,$$

and S_ψ, T_ψ are linear in each interval (ℓ_{j-1}, ℓ_j) . Then by Lemma 2.5,

$$\frac{\bar{\partial}\psi}{\partial\psi} = \sum_{k \in \mathbb{Z}} \sum_{j=1}^N \frac{\tau_j^* - 1}{\tau_j^* + 1} \chi_{f^k(A_j)} \frac{\bar{\partial}G_f}{\partial G_f},$$

where χ_E is the characteristic function, defined so that $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$.

The map ψ satisfies: $\mu_\psi|_{A_j} = k_j \overline{q_j} / |q_j|$, where $k_j = \delta_{[\phi]}^j$ and

$$q_j = \frac{\overline{\tau_j^*} - 1}{\tau_j^* + 1} \left(\frac{\partial G_f}{\partial z} dz \right)^2$$

is a holomorphic and integrable quadratic differential on A_j . By Theorem 2.7, the map $\psi|_{A_j}$ is the unique extremal map on A_j modulo the boundary ∂A_j . This implies $\delta_{[\phi]} = \max_{1 \leq j \leq N} \delta_{[\phi]}^j$. Further, if $\delta_{[\phi]}^1 = \dots = \delta_{[\phi]}^N$, then ψ is the unique extremal map.

If $\|\mu_\psi|_{A_k}\| < \delta_{[\phi]}$ for some k , then one can deform $\psi|_{A_k}$ to another map φ modulo the boundary with $\|\mu_\varphi\| \leq \delta_{[\phi]}$. This yields another quasiconformal conjugacy homotopic to ψ and with the same dilatation as ψ . \square

2.5. Appendix: The Teichmüller space of $(f, X(f))$. The general Teichmüller theory of rational maps is developed in [MS]. It's shown that the Teichmüller space of $(f, X(f))$ is isomorphic to \mathbb{H}^N . Here we precise this statement by Theorem 2.6.

We begin with the definition of the Teichmüller space of $(f, X(f))$ following [MS]. Let $QC(f, X(f))$ be the set of quasiconformal self-conjugacies of $(f, X(f))$. $QC_0(f, X(f)) \subset QC(f, X(f))$ consists of those conjugacies h isotopic to the identity in the following sense: there is a family $(h_t)_{t \in [0,1]} \subset QC(f, X(f))$ with $h_0 = id, h_1 = h$ such that $(t, z) \rightarrow (t, h_t(z))$ is a homeomorphism from $[0, 1] \times \widehat{\mathbb{C}}$ onto itself. The Teichmüller space $\text{Teich}(f, X(f))$ is the set of equivalence classes $[(g, X(g)), \phi]$ of pairs $((g, X(g)), \phi)$, where $g \in \mathcal{P}_d$, ϕ is a quasiconformal conjugacy between $(f, X(f))$ and $(g, X(g))$, and we say $((g_1, X(g_1)), \phi_1)$ and $((g_2, X(g_2)), \phi_2)$ are equivalent if there is a conformal conjugacy ψ between $(g_1, X(g_1))$ and $(g_2, X(g_2))$ such that $\phi_2^{-1} \circ \psi \circ \phi_1 \in QC_0(f, X(f))$. The Teichmüller metric between $\zeta_i = [(g_i, X(g_i)), \phi_i], i = 1, 2$ is defined by

$$d(\zeta_1, \zeta_2) = \inf \log K(\phi),$$

where the infimum is over all quasiconformal conjugacies between $(g_1, X(g_1))$ and $(g_2, X(g_2))$, homotopic to $\phi_2 \circ \phi_1^{-1}$. We remark that this metric (instead of its half) can make the map in Theorem 2.8 isometric.

Give any pair of points $a = (a_1, \dots, a_N)$, $b = (b_1, \dots, b_N) \in \mathbb{H}^N$, we define the distance $d_{\mathbb{H}^N}(a, b)$ by

$$d_{\mathbb{H}^N}(a, b) = \max_{1 \leq j \leq N} d_{\mathbb{H}}(a_j, b_j).$$

For $\tau = (\tau_1, \dots, \tau_N) \in \mathbb{H}^N$, let $\phi_\tau : \mathbb{C} \rightarrow \mathbb{C}$ solve the Beltrami equation

$$\frac{\bar{\partial}\phi_\tau}{\partial\phi_\tau} = \sum_{k \in \mathbb{Z}} \sum_{j=1}^N \frac{\tau_j - 1}{\tau_j + 1} \chi_{f^k(A_j)} \frac{\bar{\partial}G_f}{\partial G_f}.$$

The holomorphic family of quasiconformal maps is normalized so that $\phi_{(1, \dots, 1)} = id$ and $f_\tau = \phi_\tau \circ f \circ \phi_\tau^{-1} \in \mathcal{P}_d$. By the proof of Theorem 2.6, we have (compare [DP1], Lemma 5.2)

Theorem 2.8. *The map*

$$\Phi : \begin{cases} \mathbb{H}^N \rightarrow \text{Teich}(f, X(f)), \\ \tau \mapsto [(f_\tau, X(f_\tau)), \phi_\tau]. \end{cases}$$

is biholomorphic and isometric.

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